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CPT Invariance of String Models
in a Minkowski Background

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ABSTRACT

We study the space-time CPT properties of string theories formulated in a flat Minkowski background of even dimension. We define CPT as a world-sheet transformation acting on the vertex operators and we prove the CPT invariance of the string S-matrix elements. Some related issues, including the connection between spin and statistics of physical string states, are also considered.

Introduction and Summary

Recently there has been some interest in the question of possible CPT non-conservation in string theory. Indeed, some mechanisms have been proposed that would lead to CPT-breaking effects that might be detected in the next generation of experiments [1,2]. If observed, these effects could give the first experimental evidence of the existence or non-existence of strings.

Accordingly the issue of CPT invariance in string theory is quite interesting both from a theoretical and an experimental point of view. From a theoretical point of view, not too much is known and published on the space-time CPT properties of string theory. Sonoda [3] discussed and proved the space-time CPT theorem at the level of string perturbation theory for ten-dimensional heterotic strings in a Minkowski background. Kostelecky and Potting [1] proved the *dynamical* CPT invariance of the open bosonic and super string field theories, formulated in flat backgrounds –but they also suggested a method whereby CPT might be broken *spontaneously*, based on the possibility of a CPT non-invariant ground state.

For more complicated string models no general analysis has appeared. With this paper, we would therefore like to start a detailed investigation of the space-time CPT properties of general string theories. Should we always expect some kind of space-time CPT theorem to hold?

Let us start by reviewing the situation in field theory. The CPT theorem [4,5,6,7] [1] asserts that any quantum field theory is invariant under CPT transformations assuming that it satisfies the following very mild assumptions [6]:

  a) Lorentz invariance,

  b) The energy is positive definite and there exists a Poincaré-invariant vacuum, unique up to a phase factor,

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[1] For a general discussion of the CPT theorem in field theory see for example ref. [8].

c) Local commutativity, i.e. field operators at space-like separations either commute or anti-commute.

These assumptions also imply the spin-statistics theorem, i.e. fields of integer (half odd integer) spin [2] are quantized with respect to Bose (Fermi) statistics.

The proof of the CPT theorem in field theory (see ref. [6]) is actually based only on the following two assumptions:

a) Lorentz invariance,

d) spin-statistics theorem.

Turning our attention to string theory [9], we immediately see that for a generic string theory, formulated in a curved space-time background, none of the assumptions a), b) or c) is actually satisfied: There is no global concept of Lorentz invariance, potential energy in a gravitational field can be negative and, since strings are extended objects, local commutativity is not satisfied either [10,11,12]. The latter problem persists even if we restrict ourselves to a flat background. Even so, CPT invariance may still have a role to play in quantum gravity, as pointed out by Hawking [13], and also in string theory.

Clearly the first step consists in trying to define what the CPT transformation should be in a generic string theory. For non-trivial backgrounds this is likely to be a difficult problem. After all, even in a quantum field theory the CPT invariance $\phi(x) \rightarrow \phi(-x)$ can be quite obscure if we formulate the theory in a non-trivial background. Then the CPT symmetry involves not just a transformation of the quantum field but also a change of the background.

Since two-dimensional conformal field theory (CFT) is the unifying framework common to string theories in all backgrounds, the most promising approach would seem to be a world-sheet formulation of the space-time CPT transformation, where in a given string model the CPT transformation acts on the fields and states of the underlying CFT.

After having made an adequate definition of the CPT transformation as a world-sheet map, one should then make a precise formulation of the requirement of CPT invariance and proceed to look either for a proof of CPT invariance, or find a counter-example.

For a generic string theory one can then think of at least three possible situations: the CPT theorem holds true, it holds true only perturbatively, it doesn't hold at all. More

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[2] In D dimensions the concept of integer (half odd integer) spin is replaced by the concept of representations of the little group with integer (half odd integer) weights. We always use the word "spin" for convenience.

complicated cases can be imagined, but as a starting point we can limit ourselves to these three simple cases.

In this paper we will analyze the simplest class of models, that of first-quantized string models in an even-dimensional Minkowski background. Since for first-quantized string theory we do not have an off-shell (lagrangian) formulation we need to formulate and prove the CPT theorem at the level of the S-matrix elements. Since the whole scattering theory does not make much physical sense unless the S-matrix is unitary, we will always assume this to be the case for the string theories we consider.

Our strategy will be the following. We first define the space-time CPT transformation. It is a *world-sheet* transformation acting on the world-sheet operators which are the building blocks of the vertex operators.

Then, using the hypothesis of:

- explicit Lorentz invariance of the scattering amplitudes,
- validity of the spin-statistics theorem for the physical space-time spectrum,

we will formally prove that every S-matrix element in these models is invariant under the space-time CPT transformation at any loop order.

In the proof we will not need to evaluate explicitly any S-matrix element. It will be enough to know that the S-matrix elements are given by the well-known operator formulation of the Polyakov path integral; and that the world-sheet correlation functions of the physical state vertex operators satisfy the twin requirements of Lorentz invariance and spin-statistics. For this reason our results are applicable to both bosonic and type II superstring theories, as well as heterotic ones.

From this point of view, our result can even be considered to be *non-perturbative* as far as one is able to give a non perturbative definition of the Polyakov path integral, including the summation over topologies, which preserves the three assumptions listed above.

Obviously, since we have defined CPT as a transformation on the world-sheet, we need to verify that, apart from being an invariance of the S-matrix, it also has the correct space-time behaviour. We show that the CPT transformation is indeed anti-linear and transforms the vertex operator of any physical string state into a vertex operator also describing a physical string state, but with opposite spin and having the opposite sign of all gauge charges.

In this way, the CPT transformation introduces in a natural way the concept of space-time "anti-particle" in a string theory, by giving a precise world-sheet map which relates the "particle" string-state of given helicity to the "anti-particle" string state of opposite helicity.

We give some explicit examples of this in the framework of four-dimensional string models built with free world-sheet fermions [14,15,16], henceforth referred to as KLT models, since the formulation we use is that of Kawai, Lewellen and Tye [14].

Since our proof of CPT invariance is based on Lorentz invariance and spin-statistics, it becomes of interest to consider more carefully the role of the spin-statistics theorem in string theory. In field theory it is exactly in the proof of the spin-statistics theorem [17,6,18] that the assumption c) of local commutativity enters. In string theory, local commutativity does not really make sense at the first quantized level, but it does make sense at the level of string field theory, where it was found [12] that the commutator of string fields is in general non-vanishing outside the string light cone. With local commutativity violated in string theory, should we expect the spin-statistics relation to hold?

In the Neveu-Schwarz Ramond formulation of superstring theory, the spin-statistics theorem is closely related to the GSO projection [19,20,15]. It is due to the GSO projection that string states of integer space-time spin contribute to the partition function with an overall plus sign, whereas string states of half odd integer space-time spin contribute to the partition function with an overall minus sign. Actually, what we need for the proof of the CPT theorem is a seemingly stronger statement of spin-statistics, namely that any two vertex operators describing physical states of half odd integer space-time spin anti-commute whereas any other pair of physical vertex operators commute. Even if GSO by itself is not enough to guarantee this "canonical" spin-statistics relation, our expectation is that the GSO conditions are sufficient to guarantee that, by the introduction of appropriate cocycle operators, one can build vertex operators which do satisfy the "canonical" spin-statistics relation. However, we do not have any general proof of this conjecture.

As an example, we discuss the situation for a very simple (bosonized) KLT model where we find that it is indeed possible to make a cocycle choice so that the vertex operators satisfy the "canonical" spin-statistics relations. We also notice that there do exist other choices of cocycles, which lead to vertex operators not satisfying the "canonical" spin-statistics relations. Accordingly, these choices lead to theories with para-statistics, to use the terminolgy of ref. [6], which at the end should be physically equivalent to the theories

with "canonical" statistics.

The paper is organized as follows.

In section 1 we briefly review the CPT theorem in quantum field theory in various dimensions, including the special case of two-dimensional conformal field theories.

In section 2 we consider string theories in an even-dimensional Minkowski background and define the space-time CPT transformation as a world-sheet transformation acting on the vertex operators of physical string states. We then prove that our definition leads to CPT invariance of the S-matrix elements.

In section 3 we verify that the CPT transformation we have defined does indeed have the correct space-time properties when acting on physical string states, i.e. that it is an anti-linear map which changes sign on the spin and on all charges, something we further illustrate by means of two examples. We also comment on the relation between the *space-time* and *world-sheet* CPT transformations in string theory.

In section 4 we discuss various aspects of the spin-statistics theorem in string theories.

Finally, section 5 contains our concluding remarks and some open problems.

In the Appendix we give the definition of the helicity operator in four-dimensional string theory, illustrated by means of an example.

1.   The CPT theorem in D dimensional Quantum Field Theory.

In this section we review CPT transformations and the CPT theorem in even-dimensional Minkowski space-time in the context of quantum field theory.

The CPT theorem [4,5,6,7] asserts that any quantum field theory is invariant under CPT transformations assuming that it satisfies the assumptions a), b) and c) stated in the Introduction. These assumptions also imply the spin-statistics theorem, i.e. fields of integer (half odd integer) spin are quantized with respect to Bose (Fermi) statistics.

The CPT transformation actually comes in two varieties, one being the hermitean conjugate of the other:

The first, which is what Pauli [4] called *strong reflection* (SR), essentially maps a quantum field $\phi(t, \vec{x}) \rightarrow$ (phase) $\phi(-t, -\vec{x})$, i.e. it reverses time and space coordinates simultaneously. It also reverses the order of operators in an operator product and therefore

cannot be represented by any operator acting on the particle states. It is a symmetry of the operator algebra only.

The second, which we will consider to be the CPT transformation proper, is obtained by performing first SR and then the hermitean conjugation (HC). The resulting operation clearly does not change the order of operators in an operator product but instead all c-numbers are complex conjugated. It can be represented by an anti-unitary (i.e. unitary and anti-linear) operator Θ acting on the Hilbert space of physical particle states.

As is well known, the combination of the transformations of charge conjugation, parity and time reversal (C + P + T), when defined, is equal to the combination of strong reflection and hermitean conjugation (SR + HC),[3] thereby justifying the name CPT for the latter transformation.

1.1   CPT INVARIANCE IN LAGRANGIAN FIELD THEORY

It is relatively easy to demonstrate CPT invariance in the framework of Lagrangian field theory if we assume the validity of the spin-statistics theorem. Our starting point is then given by the hypothesis of

a) Lorentz invariance,

d) spin-statistics theorem.

Following Lüders [5], we start by defining the SR and CPT transformations on field operators that obey free field equations of motion and free field (anti-) commutation relations. In any (even) number D of space-time dimensions these are as follows:

For a scalar field:

$$(\partial_\mu \partial^\mu - m^2)\phi(x) = 0 \tag{1.1}$$
$$[\phi^\dagger(x), \phi(x')] = -i\Delta(x - x') \ .$$

For a spinor field:

$$(\gamma^\mu \partial_\mu + m)\psi(x) = 0 \tag{1.2}$$
$$\{\psi(x), \overline{\psi}(x')\} = i(\gamma^\mu \partial_\mu - m)\Delta(x - x') \ .$$

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[3] This is true provided that the phase for each transformation is chosen appropriately.

For a vector field:

$$(\partial_\mu \partial^\mu - m^2)\phi^\nu(x) = 0 \qquad\qquad \partial_\mu \phi^\mu(x) = 0 \qquad\qquad (1.3)$$

$$[\phi_\mu^\dagger(x), \phi_\nu(x')] = -i(\eta_{\mu\nu} - \frac{1}{m^2}\partial_\mu \partial_\nu)\Delta(x - x') \ ,$$

where our notation is to take $x = (t, \vec{x})$, the metric $\eta = \mathrm{diag}(-1, +1, \ldots, +1)$ and the gamma matrices to satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad\qquad (\gamma^\mu)^\dagger = \gamma_\mu \qquad\qquad (1.4)$$

$$\overline{\psi}(x) = \psi^\dagger(x)(-i\gamma_0) \ = \ \psi^\dagger(x)(i\gamma^0)$$

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu \qquad\qquad C^\dagger = C^{-1} \ .$$

The Schwinger function Δ is given by

$$\Delta(x) = i \int \frac{d^{D-1}\vec{p}}{(2\pi)^{D-1}2p^0} \left(e^{ip\cdot x} - e^{-ip\cdot x} \right) \quad \text{with} \quad p^0 = \sqrt{\vec{p}^2 + m^2} \qquad (1.5)$$

and clearly satisfies

$$\Delta(-x) = -\Delta(x) \ . \qquad\qquad (1.6)$$

Under SR the individual operators transform as follows

$$\phi(x) \ \xrightarrow{\text{SR}} \ \phi(-x) \qquad\qquad (1.7)$$

$$\psi(x) \ \xrightarrow{\text{SR}} \ \varphi_{\text{SR}}\gamma^{D+1}\psi(-x) \qquad \text{with} \qquad (\varphi_{\text{SR}})^2 = -(-1)^{D/2}$$

$$\overline{\psi}(x) \ \xrightarrow{\text{SR}} \ -\varphi_{\text{SR}}^*\overline{\psi}(-x)\gamma^{D+1}$$

$$\phi^\mu(x) \ \xrightarrow{\text{SR}} \ -\phi^\mu(-x) \ ,$$

where γ^{D+1} is the chirality matrix (normalized up to a sign by the requirement that the square is the unit matrix). It is easy to verify that the equations of motion and (anti-) commutation relations (1.1) – (1.3) are invariant under these transformations when we also define SR to *invert the order of the operators* and *leave c-numbers unchanged*. At this point it is obviously essential that the fields satisfy the spin-statistics relation.

For complex fields the requirement that the (anti-) commutation relations are invariant under SR only fixes the form of the SR transformations (1.7) up to an overall phase factor; but for real fields the choice of φ_{SR} is constrained by the requirement that the SR transformation is consistent with hermitean conjugation and only a sign ambiguity

remains. [4] The transformation laws given in (1.7) correspond to a choice of phase that is consistent also for real fields. The dependence on the dimension that enters into the phase φ_{SR} for the spinor field is due to the fact that the charge conjugation matrix commutes with γ^{D+1} in dimensions $D = 4k, k \in \mathbb{N}$ but anti-commutes in dimensions $D = 2 + 4k, k \in \mathbb{N}$.

For the bosonic fields the sign is chosen to agree with what we obtain by applying the tensor transformation law to the transformation $x \to -x$. Thus a field with N vector indices would transform under SR with a phase $(-1)^N$.

By definition, the CPT transformation is obtained as the SR transformation (1.7) followed by the operation of hermitean conjugation. Unlike the SR transformation, CPT may be represented by an anti-unitary operator Θ acting on the free (multi-)particle states. Each free single-particle state is completely characterized by three labels

$$|\rho\rangle = |p, \eta, \{\lambda\}\rangle , \qquad (1.8)$$

where p is the momentum of the state, η describes the spin degrees of freedom and $\{\lambda\}$ is a collective label for all charges and enumerative indices carried by the state.

In $D = 4$, we may identify η with the helicity, defined as the projection of the spin along the momentum; more precisely as the eigenvalue of the operator $H = \frac{\vec{J} \cdot \vec{p}}{|\vec{p}|}$, see also Appendix A. [5]

The CPT transformation flips the helicity and the charges, leaving the momentum invariant. More precisely

$$|\rho\rangle = |p, \eta, \{\lambda\}\rangle \xrightarrow{\text{CPT}} |\rho^{\mathrm{CPT}}\rangle = \varphi_{\mathrm{CPT}}(\eta, \{\lambda\}) |p, -\eta, \{-\lambda\}\rangle , \qquad (1.9)$$

where the phase $\varphi_{\mathrm{CPT}}(\eta, \{\lambda\})$ may depend on the quantum numbers η and $\{\lambda\}$, as indicated. Multiparticle states transform like a direct product of single-particle states:

$$|\rho_1; \rho_2; \ldots; \rho_N\rangle \xrightarrow{\text{CPT}} |\rho_1^{\mathrm{CPT}}; \rho_2^{\mathrm{CPT}}; \ldots; \rho_N^{\mathrm{CPT}}\rangle , \qquad (1.10)$$

and

$$(|\rho_1; \rho_2; \ldots; \rho_N\rangle)^{\dagger} = \langle \rho_N; \ldots; \rho_2; \rho_1| . \qquad (1.11)$$

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[4] Real (Majorana) fermions of nonzero mass can only be introduced in dimensions $D = 2 + 8k$ or $D = 4 + 8k$, $k \in \mathbb{N}$.

[5] In higher (even) dimensions, η may be thought of as the eigenvalues of the Lorentz generators $M^{12}, M^{34}, \ldots, M^{D-3,D-2}$, if the momentum points in the $(D-1)$-direction. We always use the word "helicity" for short.

Finally, the phase of the vacuum is chosen so that

$$|0\rangle \overset{\text{CPT}}{\longrightarrow} |0\rangle . \tag{1.12}$$

Having defined the SR and CPT transformations for free quantum fields (and hence for free multiparticle states) we turn our attention to an interacting theory. In this case the quantum fields are no longer free, but they can still formally be expressed in terms of the free fields by means of the time-evolution operator $U(t)$.

To show that SR remains a symmetry even in the interacting case, we must show that the interaction Lagrangian $\mathcal{L}_{\text{int}}(x)$ is invariant under the free-field transformations (1.7). To this end we note that any object carrying N Lorentz vector indices transforms under SR with a sign $(-1)^N$. For the fundamental bosonic quantum fields this holds by definition; for the derivative ∂_μ it is trivial, and by inspection it holds as well for any (pseudo-)tensor of the form $:\overline{\psi}\gamma^{\mu_1\cdots\mu_n}\psi:$ or $:\overline{\psi}\gamma^{\mu_1\cdots\mu_n}\gamma^{D+1}\psi:$. Thus, any Lorentz-invariant interaction Lagrangian that we can build by taking normal-ordered products of the fields and their derivatives will be invariant under SR.

Under CPT, the interaction Lagrangian and the S-matrix then transform according to

$$\Theta\mathcal{L}_{\text{int}}(x)\Theta^{-1} = (\mathcal{L}_{\text{int}}(-x))^\dagger \quad \text{and} \quad \Theta S\Theta^{-1} = S^\dagger . \tag{1.13}$$

Since Θ is an anti-linear operation we can consider (1.13) to express CPT invariance of the interacting theory regardless of whether the interaction Lagrangian is hermitean or not. Thus, SR invariance is considered equivalent to CPT invariance. Of course, unless the interaction Lagrangian is hermitean, the S-matrix will not be unitary and the concept of asymptotic states will be rather meaningless. For this reason we will assume unitarity to be satisfied in what follows, keeping in mind that it is strictly speaking a property distinct from that of SR/CPT invariance.

For a general vacuum expectation value of local field operators, the statement of SR/CPT invariance becomes

$$\langle 0|\Phi_1(x_1)\ldots\Phi_n(x_n)|0\rangle \tag{1.14}$$
$$\overset{\text{CPT}}{=} \langle 0|\Theta\Phi_1(x_1)\Theta^{-1}\ldots\Theta\Phi_n(x_n)\Theta^{-1}|0\rangle^*$$
$$= \langle 0|\left(\Phi_n(x_n)\right)^{\text{SR}}\ldots\left(\Phi_1(x_1)\right)^{\text{SR}}|0\rangle$$
$$= (-1)^{J_1}(-1)^{J_2}N\langle 0|\Phi_n(-x_n)\ldots\Phi_1(-x_1)|0\rangle ,$$

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where $(\Phi_i(x_i))^{\text{SR}}$ is defined by the free-field transformation laws (1.7), J_1 is the number of Lorentz vector indices, J_2 is the number of $\gamma^{D+1} = -1$ spinor indices, and

$$N = \begin{cases} i^{N_F} & \text{if} \quad D = 4k, k \in \mathbb{N} \\ (-1)^{N_\downarrow} & \text{if} \quad D = 2 + 4k, k \in \mathbb{N} \end{cases}, \tag{1.15}$$

with N_F being the total number of fermions (which is even) and $N_\downarrow$ the number of covariant spinor indices (our convention is that ψ is contravariant and $\overline{\psi}$ is covariant). The different behavior in 2 and 4 dimensions (mod 4) is again due to the different chirality properties of the charge conjugation matrix.

If the points $x_1, \ldots, x_n$ are such that all separations $x_i - x_k$, $i \neq k$, are space-like, and if we assume the validity of the spin-statistics theorem, the following condition, sometimes called *Weak Local Commutativity*, holds [6]:

$$\langle 0|\Phi_n(x_n)\ldots\Phi_1(x_1)|0\rangle = i^{N_F}\langle 0|\Phi_1(x_1)\ldots\Phi_n(x_n)|0\rangle . \tag{1.16}$$

The phase appearing is just a sign, counting how many times two fermions have been transposed in the process of reversing the order of the operators. A priori this sign is $(-1)^{N_F(N_F-1)/2}$, but since N_F is even this equals i^{N_F}.

Combining eqs. (1.14) and (1.16) we obtain another formulation of CPT invariance:

$$\langle 0|\Phi_1(x_1)\ldots\Phi_n(x_n)|0\rangle = \tag{1.17}$$
$$i^{N_F}\langle 0|\left(\Phi_1(x_1)\right)^{\text{SR}}\ldots\left(\Phi_n(x_n)\right)^{\text{SR}}|0\rangle ,$$

valid when all separations are space-like.

1.2 CPT INVARIANCE OF THE S-MATRIX ELEMENTS.

Anticipating the situation in string theory (where we have no fully-fledged Lagrangian formulation) we proceed to consider directly the S-matrix elements. Under CPT the asymptotic states $|\rho_1; \ldots; \rho_N; \substack{in \\ out}\rangle$ transform in the same way as the free multiparticle states, i.e. according to eq. (1.10), with an additional interchange of the "*in*" and "*out*" labels. In terms of the S-matrix elements the statement of field theory CPT invariance becomes

$$\langle \rho_1; \ldots; \rho_{N_{out}}; in|S|\rho_{N_{out}+1}; \ldots; \rho_N; in\rangle \tag{1.18}$$
$$\stackrel{\text{CPT}}{=} \langle \rho_1^{\text{CPT}}; \ldots; \rho_{N_{out}}^{\text{CPT}}; out|S^\dagger|\rho_{N_{out}+1}^{\text{CPT}}; \ldots; \rho_N^{\text{CPT}}; out\rangle^*$$
$$= \langle \rho_N^{CPT}; \ldots; \rho_{N_{out}+1}^{CPT}; in|S|\rho_{N_{out}}^{CPT}; \ldots; \rho_1^{CPT}; in\rangle .$$

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To mimic the situation in string theory as closely as possible we represent the S-matrix elements by means of the Lehmann-Symanzik-Zimmermann reduction formula [21] as in ref. [22]:

$$\langle \rho_1, \ldots, \rho_{N_{out}}; in|S|\rho_{N_{out}+1}, \ldots, \rho_N; in\rangle = \text{disconnected terms} + \quad (1.19)$$

$$\left(\prod_{j=1}^{N} \frac{i}{\sqrt{Z_j}}\right) \int \left(\prod_{j=1}^{N} \mathrm{d}^4 x_j\right) \langle 0|\mathrm{T}V_{\langle\rho_1|}(x_1) \ldots V_{|\rho_N\rangle}(x_N)|0\rangle \ ,$$

where we have a Field Theory Vertex (FTV) $V_{|\rho\rangle}(x)$ corresponding to the 1-particle ket-state $|\rho; in\rangle$ and similarly a FTV $V_{\langle\rho|}(x) = (V_{|\rho\rangle}(x))^\dagger$ corresponding to the 1-particle bra-state $\langle\rho; in|$.

Using the identity (1.14) on the correlation function appearing in (1.19) one finds that the statement of CPT (or SR) invariance can be formulated as follows [6]

$$\langle \rho_1; \ldots; \rho_{N_{out}}|S|\rho_{N_{out}+1}; \ldots; \rho_N\rangle = \quad (1.20)$$

$$\text{disconnected terms} + \left(\prod_{j=1}^{N} \frac{i}{\sqrt{Z_j}}\right) \int \left(\prod_{j=1}^{N} \mathrm{d}^4 x_j\right) \times$$

$$\langle 0|\mathrm{T}\left(V_{|\rho_N\rangle}(x_N)\right)^{SR} \ldots \left(V_{\langle\rho_1|}(x_1)\right)^{SR}|0\rangle \ .$$

On the other hand, we may also use eq. (1.19) to rewrite

$$\langle \rho_N^{\mathrm{CPT}}; \ldots; \rho_{N_{out}+1}^{\mathrm{CPT}}|S|\rho_{N_{out}}^{\mathrm{CPT}}; \ldots; \rho_1^{\mathrm{CPT}}\rangle = \quad (1.21)$$

$$\text{disconnected terms} + \left(\prod_{j=1}^{N} \frac{i}{\sqrt{Z_j}}\right) \int \left(\prod_{j=1}^{N} \mathrm{d}^4 x_j\right) \times$$

$$\langle 0|\mathrm{T}V_{\langle\rho_N^{\mathrm{CPT}}|}(x_N) \ldots V_{|\rho_1^{\mathrm{CPT}}\rangle}(x_1)|0\rangle \ .$$

Since the S-matrix element appearing on the left-hand side of eq. (1.20) equals the one appearing on the left-hand side of eq. (1.21) (by the identity (1.18)) it follows that the right-hand sides of these two equations must coincide as well. Since this is true for any combination of states, it follows that

$$\left(V_{|\rho\rangle}(x)\right)^{SR} = V_{\langle\rho^{\mathrm{CPT}}|}(-x) = \left(V_{|\rho^{\mathrm{CPT}}\rangle}(-x)\right)^\dagger \ , \quad (1.22)$$

which can be viewed as an indirect way of defining the CPT transformed one-particle state $|\rho^{\mathrm{CPT}}\rangle$ directly from the known transformation properties under SR of the FTV $V_{|\rho\rangle}$. This point of view will turn out to be very useful for our formulation of the CPT transformation in string theory.

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[6] Since we will mainly consider states of the "*in*" variety, unless otherwise stated $|\rho\rangle = |\rho; in\rangle$, i.e. we drop the "*in*" label.

1.3 THE CPT THEOREM FOR TWO-DIMENSIONAL CONFORMAL FIELD THEORY.

For the purpose of string theory, we are interested in world-sheets with a two-dimensional metric, a compactified space-direction and possibly a non-trivial topology. Since this is incompatible with 2-dimensional Lorentz invariance, the CPT theorem in its standard form, as described in subsection 1.1, is not directly applicable.

However, we are only interested in 2-dimensional *conformal* field theories, where the role of the SR transformation is naturally taken by the Belavin-Polyakov-Zamolodchikov (BPZ) transformation [23]

$$z \to w = \frac{1}{z} \; , \tag{1.23}$$

which defines a globally conformal diffeomorfism on the sphere. In terms of the cylindrical coordinates (τ, σ) we have $z = \exp\{\tau + i\sigma\}$ and the BPZ transformation (1.23) is seen to map $(\tau, \sigma) \to (-\tau, -\sigma)$, as desired. A primary conformal field $\Phi_{(\Delta,\overline{\Delta})}$ of conformal dimension $(\Delta, \overline{\Delta})$ transforms as

$$\begin{aligned}
\Phi_{(\Delta,\overline{\Delta})}(z = \zeta, \bar{z} = \bar{\zeta}) \; &\overset{\mathrm{BPZ}}{\longrightarrow} \; \Phi_{(\Delta,\overline{\Delta})}(w = \zeta, \bar{w} = \bar{\zeta}) \\
&= \; (-1)^{\Delta - \overline{\Delta}} \left(\frac{1}{\zeta^2}\right)^{\Delta} \left(\frac{1}{\bar{\zeta}^2}\right)^{\overline{\Delta}} \Phi_{(\Delta,\overline{\Delta})}(z = 1/\zeta, \bar{z} = 1/\bar{\zeta}) \; ,
\end{aligned} \tag{1.24}$$

which involves a choice of phase whenever $\Delta - \overline{\Delta}$ is not an integer. Conformal invariance implies the identity [7]

$$\langle \; \Phi_{\Delta_1}(z = \zeta_1) \ldots \Phi_{\Delta_n}(z = \zeta_n) \; \rangle \; = \; \langle \; \Phi_{\Delta_1}(w = \zeta_1) \ldots \Phi_{\Delta_n}(w = \zeta_n) \; \rangle \; . \tag{1.25}$$

On the sphere it is also possible to introduce the operation of hermitean conjugation, for example at the level of the mode operators. We define

$$(\Phi_\Delta(z = \zeta))^\dagger \; = \; \left(\frac{1}{\zeta^*}\right)^{2\Delta} \widehat{\Phi}_\Delta(z = 1/\zeta^*) \; , \tag{1.26}$$

where $\widehat{\Phi}_\Delta$ is a primary conformal field of the same dimension as Φ_Δ. The field Φ_Δ is said to be hermitean if $\Phi_\Delta = \widehat{\Phi}_\Delta$ and anti-hermitean if $\Phi_\Delta = -\widehat{\Phi}_\Delta$. The peculiar behaviour

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[7] For notational convenience we restrict ourselves to chiral conformal fields in the rest of this subsection.

of the argument in eq. (1.26) is due to the fact that we are considering imaginary time on the world-sheet. Rotating back to real time, we have $z = \zeta = \exp\{i(\tau + \sigma)\}$ and $1/\zeta^* = \zeta$.

By means of the hermitean conjugate, eq. (1.25) can be reformulated as follows

$$
\begin{aligned}
\langle\, \Phi_{\Delta_1}(z = \zeta_1) \ldots \Phi_{\Delta_n}(z = \zeta_n)\, \rangle &= \langle\, (\Phi_{\Delta_n}(w = \zeta_n))^\dagger \ldots (\Phi_{\Delta_1}(w = \zeta_1))^\dagger\, \rangle^* \\
&= (-1)^{\Delta_1} \ldots (-1)^{\Delta_n}\, \langle\, \widehat{\Phi}_{\Delta_n}(z = \zeta_n^*) \ldots \widehat{\Phi}_{\Delta_1}(z = \zeta_1^*)\, \rangle^*\,,
\end{aligned}
\tag{1.27}
$$

and unlike the individual transformations of BPZ and hermitean conjugation, which are defined on the sphere, the identity between the first and the last correlator in eq. (1.27) actually holds at *any* genus for a *unitary* conformal field theory [3,24,25] and is referred to as CPT invariance in two dimensions. It expresses the symmetry between a given Riemann surface and its anti-holomorphic "mirror image". It differs from the standard formulation (1.14) of CPT invariance in having the opposite ordering of the operators appearing in the complex conjugated correlator. As such it is more akin to the relation (1.17) expressing the combination of CPT invariance and Weak Local Commutativity.

Along similar lines it is tempting to define a two-dimensional (world-sheet) CPT transformation by

$$
\Phi_\Delta(z = \zeta) \overset{\text{WS-CPT}}{\longrightarrow} (\Phi_\Delta(w = \zeta))^\dagger = (-1)^{-\Delta}\widehat{\Phi}_\Delta(z = \zeta^*)
\tag{1.28}
$$

but one should keep in mind that this is only a substitution rule, rather than a genuine CPT transformation, inasmuch as it involves an inversion of the ordering of operators and therefore cannot be represented by any operator acting on states.

For non-chiral conformal fields the phase in eq. (1.28) is replaced by $(-1)^{\overline{\Delta}-\Delta}$.

2. The space-time CPT theorem for heterotic strings in a Minkowski background

In this section we consider first-quantized heterotic string models in a D-dimensional Minkowski background (D even), having a unitary S-matrix and satisfying Lorentz-invariance as well as the space-time spin-statistics theorem in a form that we will formulate more precisely below. As we pointed out already in the Introduction, and saw in more detail in section 1, these assumptions suffice to prove the CPT theorem in quantum field theory. We will show that this is also the case in string theory. Our procedure is to

first define an SR-like transformation on the fields defined on the world-sheet, which is a symmetry of the underlying 2-dimensional conformal field theory; next, we construct the space-time CPT transformation on the set of physical string states by means of the SR symmetry and finally we prove that with this definition of CPT conjugate string states, the identity (1.18) holds for the string S-matrix elements.

To describe the heterotic superstring we use the Neveu-Schwarz Ramond (NSR) formalism but it will become obvious that our results actually hold more generally, for example also for string models in the Green-Schwarz formalism. In the NSR formalism we have various free conformal fields: The space-time coordinates X^μ, their chiral world-sheet superpartners ψ^μ, the reparametrization ghosts b, c and $\bar{b}, \bar{c}$, and the superghosts β, γ. On top of this we have various internal degrees of freedom described by an "internal" CFT with left-moving (right-moving) central charge 22 (9). These may or may not be free.

2.1   THE SPACE-TIME SR TRANSFORMATION

We define the space-time SR transformation in the string CFT in the obvious way, taking

$$X^\mu \xrightarrow{\text{SR}} -X^\mu \,, \qquad\qquad \psi^\mu \xrightarrow{\text{SR}} -\psi^\mu \,, \tag{2.1}$$

with the (super) ghosts, as well as all fields pertaining to the "internal" CFT, remaining unchanged. The SR transformation (2.1) is *linear* (i.e. it does not complex conjugate *c*-numbers); unlike its quantum field theory namesake *it does not invert the order of operators*. This convention is seen to be the sensible one, since only thus defined will the SR transformation leave invariant the world-sheet action of the string theory, the BRST current, and the world-sheet fermion number operators obtained from the currents $\psi^0\psi^1$ and $i\psi^2\psi^3$, and hence the GSO projection conditions.

Next we define how SR acts on the vacuum (or vacua) of any given sector of the string theory. There are two kinds of sectors. Those containing states of integer space-time spin, where the vacuum transforms as a Lorentz singlet, and those containing states of half odd integer space-time spin, where it transforms as a Lorentz spinor. The vacuum may also be parametrized by various other quantities $\{\lambda\}$, pertaining to the "internal" CFT; from the space-time point of view these quantities are interpreted as charges and various labels. We define

$$|0; \{\lambda\}\rangle \xrightarrow{\text{SR}} |0; \{\lambda\}\rangle \qquad \text{for a sector of integer spin,} \tag{2.2}$$

$$|\alpha; \{\lambda\}\rangle \xrightarrow{\text{SR}} \varphi_{\text{SR}} \, (\gamma^{D+1})_\alpha^{\ \beta} \, |\beta; \{\lambda\}\rangle \qquad \text{for a sector of half odd integer spin,}$$

where φ_{SR} is some as yet unspecified phase factor. The presence of the γ^{D+1} is needed in order to obtain an unambiguous result for the SR transform of the state

$$\psi_0^\mu \, |\alpha; \{\lambda\}\rangle = \frac{1}{\sqrt{2}} (\gamma^\mu)_\alpha^{\ \beta} \, |\beta; \{\lambda\}\rangle \, , \qquad (2.3)$$

where ψ_0^μ is the zero-mode of the field ψ^μ. We may introduce spin fields that create the various vacua from the conformal one; collecting all indices of the vacuum into a single one, $\mathbb{A}$, we may write

$$S_{\mathbb{A}}(z = \bar{z} = 0)|0\rangle \equiv |\mathbb{A}\rangle \, . \qquad (2.4)$$

In terms of the spin fields the transformation laws (2.2) become

$$S_{\mathbb{A}} \xrightarrow{\text{SR}} S_{\mathbb{A}} \qquad \text{for a sector of integer spin,} \qquad (2.5)$$

$$S_{\mathbb{A}} \xrightarrow{\text{SR}} \varphi_{\text{SR}} \left(\boldsymbol{\Gamma}^{D+1}\right)_{\mathbb{A}}^{\ \mathbb{B}} S_{\mathbb{B}} \qquad \text{for a sector of half odd integer spin,}$$

where obviously, writing $\mathbb{A} = (\alpha; \{\lambda\})$ and $\mathbb{B} = (\beta; \{\lambda'\})$, we have defined

$$\left(\boldsymbol{\Gamma}^{D+1}\right)_{\mathbb{A}}^{\ \mathbb{B}} = (\gamma^{D+1})_\alpha^{\ \beta} \, \delta_{\{\lambda\}}^{\{\lambda'\}} \, . \qquad (2.6)$$

The phase factor φ_{SR} is fixed up to a sign by the requirement that SR should commute with hermitean conjugation. Indeed, if we bosonize the fermions ψ^μ, introducing $D/2$ scalar fields $\phi_{(0)}, \phi_{(1)}, \ldots, \phi_{(D/2-1)}$, the spin field operator $S_{\mathbb{A}}$ pertaining to a sector of half odd integer spin may be written as

$$S_{\mathbb{A}} = e^{a_0 \phi_{(0)}} e^{a_1 \phi_{(1)}} \ldots e^{a_{D/2-1} \phi_{(D/2-1)}} \times \text{(internal part)} \times \text{(cocycle factor)} \, , \qquad (2.7)$$

where all the quantities $a_0, a_1, \ldots$ take values $\pm 1/2$ and the space-time spinor index $\alpha = (a_0, a_1, \ldots, a_{D/2-1})$. In this formulation the matrix γ^{D+1} may be thought of as a direct product of $D/2$ matrices $\sigma_3^{(i)}$, which take values $+1$ (-1) depending on whether the number $a_i = +1/2$ $(-1/2)$.

When the bosonization is performed in Minkowski space-time, the operator field $\phi_{(0)}$ is hermitean, whereas the other ones, $\phi_{(1)}, \ldots, \phi_{(D/2-1)}$, are anti-hermitean [26]. Therefore, when we take the hermitean conjugate of the spin field, we obtain a new spin field, where the sign has changed on all the a_i except a_0. Accordingly, the value of γ^{D+1} has changed by the sign $(-1)^{D/2-1}$.

Therefore, the requirement that the operations of SR and hermitean conjugation should commute leads to the constraint

$$\varphi_{\text{SR}}^* = \varphi_{\text{SR}}(-1)^{D/2-1} \tag{2.8}$$

or equivalently

$$(\varphi_{\text{SR}})^2 = -(-1)^{D/2} \, , \tag{2.9}$$

which agrees with the phase choice found in field theory when one requires compatibility of SR with a Majorana reality condition (q.v. eq. (1.7)).

In the following we will always assume the phase φ_{SR} to be given in accordance with (2.9).

The SR transformation is a symmetry of the conformal field theory describing the space-time fields X^μ, since it leaves both the action and the path-integral measure invariant. Therefore we have the identity

$$\langle (\mathcal{O}_1)^{\text{SR}}(z_1, \bar{z}_1) \dots (\mathcal{O}_N)^{\text{SR}}(z_N, \bar{z}_N) \rangle = \langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_N(z_N, \bar{z}_N) \rangle \, , \tag{2.10}$$

which holds for any correlation function in the CFT of the fields X^μ.

If we proceed to consider the CFT of all the other fields in the string theory, including the fields ψ^μ, together with the "internal" CFT and the (super) ghosts, the SR transformation is again clearly a symmetry of the action. However, when the fields ψ^μ contain branch cuts (corresponding to the presence of pairs of spin fields transforming as space-time spinors), the path integral measure is no longer SR invariant. Instead we obtain the modified SR identity

$$\langle (\mathcal{O}_1)^{\text{SR}}(z_1, \bar{z}_1) \dots (\mathcal{O}_N)^{\text{SR}}(z_N, \bar{z}_N) \rangle = (-1)^{N_{\text{FP}}} \langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_N(z_N, \bar{z}_N) \rangle \, , \tag{2.11}$$

where $2N_{\text{FP}}$ is the number of space-time spinorial spin fields. [8]

To prove this, we do not have to specify in detail how the path-integral measure is defined. It is sufficient to assume that it is defined to be invariant under Lorentz transformations continuously connected to the identity.

---

[8] That the number of space-time spinorial spin-fields is even for nonzero correlators is a consequence of the assumption of space-time Lorentz invariance. No invariant tensor exists with an odd number of spinor indices.

Indeed, consider –in the operator formulation– the correlation function involving $2N_{\text{FP}}$ fields

$$\left(\mathcal{O}_i^F\right)^{\mu_1^{(i)}\ldots\mu_{n_i}^{(i)}}_{\alpha_i} \qquad i = 1, \ldots, 2N_{\text{FP}} \tag{2.12}$$

that carry a space-time spinor index α_i, and N_B operator fields

$$\left(\mathcal{O}_i^B\right)^{\nu_1^{(i)}\ldots\nu_{m_i}^{(i)}} \qquad i = 1, \ldots, N_{\text{B}} \tag{2.13}$$

that do not. The fields may also carry various vector indices, as shown. They will in general also carry all sorts of indices connected with the "internal" CFT, but these are all suppressed as they do not play any role in the present discussion. We imagine all Lorentz indices in (2.12) and (2.13) to be carried by the world-sheet operators; no Dirac spinors, polarization or momentum vectors etc. have been introduced. Being c-numbers, they do not transform under SR anyway.

The assumption of invariance of the path integral under infinitesimal Lorentz transformations implies first, that the correlation function of the operators (2.12) and (2.13) transforms under such transformations in accordance with the index structure of the operators, i.e. as an object with $2N_{\text{FP}}$ covariant spinor indices and $n_1+\ldots+n_{2N_{\text{FP}}}+m_1+\ldots+m_{N_B} \equiv N_V$ contravariant vector indices, and second, that the correlation function can be expressed only by means of the invariant tensors at our disposal

$$\eta_{\mu\nu}, \ \eta^{\mu\nu}, \ \epsilon^{\mu_1\mu_2\ldots\mu_D}, \ (\gamma^\mu)_\alpha^{\ \beta}, \ C_{\alpha\beta} \ , \tag{2.14}$$

where $C_{\alpha\beta}$ is the spinor metric. We do not need to include γ^{D+1} explicitly in the list, since $\gamma^{D+1} = \frac{1}{D!}\varphi_{\text{SR}}\epsilon_{\mu_1\mu_2\ldots\mu_D}\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_D}$.[9] Also notice that for D even there exist two spinor metrics, but one is obtained from the other merely by multiplying with γ^{D+1}. We may remove this ambiguity by taking C to be the charge conjugation matrix, i.e. satisfying

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu \ . \tag{2.15}$$

Finally notice that we do not need to include the inverse spinor metric, because the only non-trivial way this could possibly enter is with both indices contracted with gamma matrices, so that C^{-1} is multiplied from the right by a gamma matrix and from the left by a transposed gamma matrix. But using eq. (2.15) we may always move any such factor

---

[9] Since $(\varphi_{\text{SR}})^2 = (-1)^{D/2-1}$, up to a sign this is the usual definition of γ^{D+1} in (even) D-dimensional Minkowski space-time.

of C^{-1} to the left, until we reach the end of the string of gamma matrices; and since all upper spinor indices are contracted, it will then inevitably cancel against a factor of C.

Having made these observations it is clear that to get the right number of lower spinor indices our correlator must contain exactly N_{FP} factors of the spinor metric. Each spinor metric may be multiplied by gamma matrices from the left and by transposed gamma matrices from the right. By repeated use of eq. (2.15) we may bring C to the right, "untransposing" all gamma-matrices in the process. This way we end up with N_{FP} structures of the type

$$(\gamma^{\mu_1} \ldots \gamma^{\mu_n} C)_{\alpha\beta} \ . \tag{2.16}$$

Now, under SR each of the operators (2.12) picks up a factor $(-1)^{n_i} \varphi_{\text{SR}} \gamma^{D+1}$ (acting on the spinor index α_i), while the operators (2.13) all pick up the factor $(-1)^{m_i}$. Thus, the N_V vector indices give rise to the sign $(-1)^{N_V}$. Since

$$
\begin{aligned}
(\varphi_{\text{SR}})^2 \gamma^{D+1} \gamma^{\mu_1} \ldots \gamma^{\mu_n} C \gamma^{D+1} &= (\varphi_{\text{SR}})^2 (-1)^n (-1)^{D/2} \gamma^{\mu_1} \ldots \gamma^{\mu_n} C \\
&= -(-1)^n \gamma^{\mu_1} \ldots \gamma^{\mu_n} C \ ,
\end{aligned}
\tag{2.17}
$$

the N_{FP} structures of the type (2.16) give rise to a further sign which is just $(-1)^{N_{\text{FP}}}$, times $(-1)^{N'_G}$, where N'_G is the number of gamma matrices that appear in the expression for the correlator inside structures of the type (2.16). Lorentz invariance only allows gamma matrices to appear without an accompanying spinor metric in one way, namely as traces $(\gamma^{\mu_1} \ldots \gamma^{\mu_n})_\alpha{}^\alpha$, but these are only nonzero if n is even, so we may write $(-1)^{N'_G} = (-1)^{N_G}$, where N_G is the total number of gamma matrices appearing. Since the metric and the Levi-Civita tensor always carry an even number of Lorentz vector indices, we may in fact write $(-1)^{N'_G} = (-1)^{N_G} = (-1)^{N_V^{\text{tot}}}$, where N_V^{tot} is the total number of Lorentz vector indices appearing in the expression for the correlator, counting both covariant and contravariant indices. Some of these indices will in general be summed over. It is clear that $N_V^{\text{tot}} = N_V + N_D$, where N_D is the number of dummy indices, which is even, since any dummy index appears exactly twice. Therefore $(-1)^{N_V} = (-1)^{N_G} = (-1)^{N'_G}$, and we find that SR transforms the correlator into itself times a phase factor that finally reduces to $(-1)^{N_{\text{FP}}}$.

This concludes our proof of eq. (2.11). A priori this identity holds for the CFT *excluding* the fields X^μ, but since the CFT of the latter satisfies the identity (2.10), it is clear that eq. (2.11) actually holds for the entire CFT underlying the string theory in question.

In summary, the SR transformation we have defined can be considered to act on the space-time indices carried by the world-sheet fields and the identity (2.11) follows from the one basic assumption that the string models we consider are invariant under space-time Lorentz transformations continuously connected to the identity. In the following two subsections we show how the SR transformation can be used to define a CPT transformation on the space of physical string states and to prove the invariance of the S-matrix under this transformation.

2.2 THE SPACE-TIME CPT TRANSFORMATION

Having introduced the space-time SR transformation we now turn our attention to the definition of the space-time CPT transformation. But first we need to introduce some very general notation for the string S-matrix elements.

We define the T-matrix element (loosely referred to as "the amplitude") as the connected S-matrix element with certain normalization factors removed

$$\frac{\langle \rho_1, \ldots, \rho_{N_{\text{out}}} | S | \rho_{N_{\text{out}}+1}, \ldots, \rho_N \rangle_{\text{connected}}}{\prod_{i=1}^{N} \left(\langle \rho_i | \rho_i \rangle \right)^{1/2}} = \tag{2.18}$$

$$i(2\pi)^D \delta^D(p_1 + \ldots p_{N_{\text{out}}} - p_{N_{\text{out}}+1} - \ldots - p_N) \prod_{i=1}^{N} (2p_i^0 V)^{-1/2} \times$$

$$T(\rho_1; \ldots; \rho_{N_{\text{out}}} | \rho_{N_{\text{out}}+1}; \ldots; \rho_N) \,,$$

where p_i is the momentum of the i'th string state, all of them having $p_i^0 > 0$, and V is the usual volume-of-the-world factor. We also introduce the dimensionless momentum $k_\mu \equiv \sqrt{\frac{\alpha'}{2}} p_\mu$. The Minkowski metric is $\text{diag}(-1, 1, \ldots, 1)$.

Corresponding to each on-shell single-string state $|\rho\rangle$ appearing in the T-matrix element we have a BRST-invariant vertex operator $\mathcal{W}_{|\rho\rangle}(z, \bar{z})$, defined by

$$|\rho\rangle = \lim_{\zeta, \bar{\zeta} \to 0} \mathcal{W}_{|\rho\rangle}(z = \zeta, \bar{z} = \bar{\zeta}) |0\rangle \,, \tag{2.19}$$

which is a primary conformal field of dimension $\Delta = \overline{\Delta} = 0$.

Similarly, as discussed in more details in ref. [22], to each state $\langle \rho |$ we associate a vertex operator $\mathcal{W}_{\langle \rho |}(z, \bar{z})$, which –apart from a phase factor χ, depending on the superghost charge (i.e. the picture) of the state $|\rho\rangle$– is simply obtained from the vertex operator $\mathcal{W}_{|\rho\rangle}$

by means of the world-sheet CPT transformation (1.28):

$$\mathcal{W}_{\langle\rho|}(z = \zeta, \bar{z} = \bar{\zeta}) = \chi \left(\mathcal{W}_{|\rho\rangle}(z = \zeta^*, \bar{z} = \bar{\zeta}^*)\right)^{\mathrm{WS-CPT}} \tag{2.20}$$

$$= \chi \, \widehat{\mathcal{W}}_{|\rho\rangle}(z = \zeta, \bar{z} = \bar{\zeta}) \; .$$

Whereas the operator $\mathcal{W}_{|\rho\rangle}$ creates a ket state of positive energy, i.e. is proportional to $e^{ik\cdot X}$ with $k^0 > 0$, the operator $\mathcal{W}_{\langle\rho|}$ is proportional to $e^{-ik\cdot X}$ and thus creates a ket state with negative energy.

As it is well known, the T-matrix element is given by the formula

$$T(\rho_1; \ldots; \rho_{N_{\mathrm{out}}} | \rho_{N_{\mathrm{out}}+1}; \ldots; \rho_N) = \sum_{g=0}^{\infty} C_g \int \mathrm{d}\mu \sum_{\substack{\mathrm{spin} \\ \mathrm{structures}}} \tag{2.21}$$

$$\langle\langle \begin{pmatrix} \mathrm{ghost} \\ \mathrm{insertions} \end{pmatrix} \begin{pmatrix} \mathrm{picture\ changing} \\ \mathrm{operators} \end{pmatrix} \mathcal{W}_{\langle\rho_1|}(z_1, \bar{z}_1) \ldots \mathcal{W}_{|\rho_N\rangle}(z_N, \bar{z}_N) \rangle\rangle \; ,$$

which involves a formal sum over topologies (plus, in the NSR formulation of superstring theory, a summation over spin structures), and an integral over moduli, where the integrand is given by the correlator of the vertex operators, with various ghost and picture changing operators inserted. For what follows we do not need to specify the details in the formula (2.21), which anyway depend on which kind of string theory we consider. The exact form taken by this formula in the specific instance of heterotic string theory can be found for example in ref. [22].

We are now ready to define the space-time CPT transformation in string theory. At the level of the vertex operators we define, in close analogy with eq. (1.22),

$$\left(\mathcal{W}_{|\rho\rangle}\right)^{\mathrm{SR}} \equiv \mathcal{W}_{\langle\rho^{\mathrm{CPT}}|} \; . \tag{2.22}$$

This definition makes sense: The vertex operator $\mathcal{W}_{|\rho\rangle}$ has positive energy; the SR transformed operator therefore has negative energy ($\exp\{ik \cdot X\} \xrightarrow{\mathrm{SR}} \exp\{-ik \cdot X\}$) and may be thought of as the vertex operator pertaining to an outgoing string state $\langle\rho^{\mathrm{CPT}}|$. To find the state $|\rho^{\mathrm{CPT}}\rangle$ or, equivalently, the operator $\mathcal{W}_{|\rho^{\mathrm{CPT}}\rangle}$, we simply use the map (2.20) "backwards", to obtain

$$\mathcal{W}_{|\rho\rangle} \xrightarrow{\mathrm{CPT}} \mathcal{W}_{|\rho^{\mathrm{CPT}}\rangle} = \chi \, \widehat{\mathcal{W}}_{\langle\rho^{\mathrm{CPT}}|} = \chi \left(\widehat{\mathcal{W}}_{|\rho\rangle}\right)^{\mathrm{SR}} = \left(\mathcal{W}_{\langle\rho|}\right)^{\mathrm{SR}} \; . \tag{2.23}$$

Thus, the *space-time* CPT transformation mapping $\mathcal{W}_{|\rho\rangle}$ into $\mathcal{W}_{|\rho^{\mathrm{CPT}}\rangle}$ is essentially the combination of the SR transformation defined by eqs. (2.1) and (2.5) with the *world-sheet*

CPT transformation (1.28) which relates $\mathcal{W}_{|\rho\rangle}$ to $\mathcal{W}_{\langle\rho|}$, as in eq. (2.20). Notice that since hermitean conjugation and SR have been arranged to commute (by the phase choice (2.9)) no ambiguity arises in the definition (2.23) by interchanging the order of the SR and world-sheet CPT transformations.

The transformation $|\rho\rangle \xrightarrow{\text{CPT}} |\rho^{\text{CPT}}\rangle$ that we have defined is clearly anti-linear, due to the presence of hermitean conjugation in the definition (1.26) of the "hatted" operator. As was shown in ref. [22] the operator $\mathcal{W}_{\langle\rho|}$ satisfies the requirement of BRST invariance and the GSO conditions if and only if $\mathcal{W}_{|\rho\rangle}$ does. Since the SR transformation defined in the previous subsection leaves the BRST current and the GSO conditions invariant, the conclusion is that the state $|\rho^{\text{CPT}}\rangle$, defined by eq. (2.23), is BRST invariant and in the GSO projected spectrum if and only if $|\rho\rangle$ is.

2.3   PROOF OF THE SPACE-TIME CPT THEOREM

We are now ready to prove the space-time CPT theorem in string theory. The ingredients will be Lorentz invariance, as encoded in the identity (2.11), together with *space-time spin-statistics*, as expressed by the following basic assumption:

For any pair of physical, BRST-invariant vertex operators $\mathcal{W}_{|\rho_1\rangle}$ and $\mathcal{W}_{|\rho_2\rangle}$, describing incoming string states in the GSO projected spectrum that can appear together in a nonzero amplitude, it is required that

$$\mathcal{W}_{|\rho_1\rangle}(z_1, \bar{z}_1)\mathcal{W}_{|\rho_2\rangle}(z_2, \bar{z}_2) \;=\; \pm \mathcal{W}_{|\rho_2\rangle}(z_2, \bar{z}_2)\mathcal{W}_{|\rho_1\rangle}(z_1, \bar{z}_1) \;, \tag{2.24}$$

the sign being minus if both $|\rho_1\rangle$ and $|\rho_2\rangle$ carry half odd integer space-time spin and plus otherwise.

By the relation (2.20) it is clear that also the operators $\mathcal{W}_{\langle\rho_1|}$ and $\mathcal{W}_{\langle\rho_2|}$, or $\mathcal{W}_{|\rho_1\rangle}$ and $\mathcal{W}_{\langle\rho_2|}$, will then satisfy Bose or Fermi statistics on the world-sheet depending on whether their space-time spin is integer or half odd integer.

At the level of the amplitudes given by eq. (2.21) our assumption (2.24) ensures that

$$T(\rho_1; \ldots; \rho_{N_{\text{out}}} | \rho_{N_{\text{out}}+1}; \ldots; \rho_i; \rho_{i+1}; \ldots; \rho_N) \;= \tag{2.25}$$
$$\pm\, T(\rho_1; \ldots; \rho_{N_{\text{out}}} | \rho_{N_{\text{out}}+1}; \ldots; \rho_{i+1}; \rho_i; \ldots; \rho_N)$$

(and similarly for interchange of outgoing string states), with the sign being minus if the single-string states $|\rho_i\rangle$ and $|\rho_{i+1}\rangle$ both describe string states with half odd integer space-time spin, and plus otherwise.

It is now straightforward to prove the space-time CPT invariance (1.18) of the string S-matrix elements at any loop order:

$$
\begin{aligned}
T(\rho_N^{\mathrm{CPT}}; \ldots; \rho_{N_{\mathrm{out}}+1}^{\mathrm{CPT}} | \rho_{N_{\mathrm{out}}}^{\mathrm{CPT}}; \ldots; \rho_1^{\mathrm{CPT}}) \ &= \hspace{4cm} (2.26) \\
&= \sum \int \langle\langle (\ldots) \mathcal{W}_{\langle \rho_N^{\mathrm{CPT}}|} \ldots \mathcal{W}_{|\rho_1^{\mathrm{CPT}}\rangle} \rangle\rangle \\
&= \sum \int \langle\langle (\ldots)^{\mathrm{SR}} \left(\mathcal{W}_{|\rho_N\rangle} \right)^{\mathrm{SR}} \ldots \left(\mathcal{W}_{\langle\rho_1|} \right)^{\mathrm{SR}} \rangle\rangle \\
&= (-1)^{N_{\mathrm{FP}}} \sum \int \langle\langle (\ldots) \mathcal{W}_{|\rho_N\rangle} \ldots \mathcal{W}_{\langle\rho_1|} \rangle\rangle \\
&= (-1)^{N_{\mathrm{FP}}} (-1)^{N_{\mathrm{FP}}(2N_{\mathrm{FP}}-1)} \sum \int \langle\langle (\ldots) \mathcal{W}_{\langle\rho_1|} \ldots \mathcal{W}_{|\rho_N\rangle} \rangle\rangle \\
&= T(\rho_1; \ldots; \rho_{N_{\mathrm{out}}} | \rho_{N_{\mathrm{out}}+1}; \ldots; \rho_N) \, .
\end{aligned}
$$

In going from the second to the third line we used the definition (2.22) of the CPT transformation (and the inverse relation (2.23)), together with the fact that the various ghost and picture changing operators $(\ldots)$ are SR invariant; Next we used the SR identity (2.11), which followed from the assumption of Lorentz invariance; Finally, we used the spin-statistics assumption (2.24) to invert the order of the vertex operators. Inverting the order of the $2N_{\mathrm{FP}}$ vertex operators of half odd integer spin produces the sign $(-1)^{N_{\mathrm{FP}}(2N_{\mathrm{FP}}-1)}$ which cancels the sign appearing in the SR identity (2.11).

This concludes the proof of the space-time CPT theorem.

2.4 Comments on the proof of the space-time CPT theorem

As we have already noticed, the SR transformation, defined in string theory by eqs. (2.1), (2.5) and (2.9), is sensitive only to the Lorentz spacetime indices carried by the world-sheet operators. As such it does not depend on the details of the explicit string model under consideration and is of very general application.

Also the assumptions used in the proof of the spacetime CPT theorem are very general. We considered a D-dimensional (D even) string theory on a Minkowski background and we required

- unitarity of the S-matrix,
- explicit Lorentz invariance of the string theory,
- validity of the space-time spin-statistics theorem (2.24) for the physical string state spectrum.

The assumption of unitarity of the S-matrix does not play any direct role in the proof of the space-time CPT theorem as we have formulated it. Indeed, as we pointed out in section 1, the question of space-time CPT invariance is strictly speaking independent of the assumption of unitarity. But without unitarity, the S-matrix would of course make little physical sense and most of our statements would be valid only at the formal level.

We also note that, at the level of the string amplitudes, CPT invariance and unitarity appear in rather different ways. As we have seen, CPT invariance is implemented at the level of the vertex operator correlation function, i.e. is an invariance of the modular integrand, q.v. eq. (2.26). By contrast, unitarity only appears *after* integrating over the moduli, with appropriate regularizations of short-distance divergencies [27,28,29,30].

To illustrate this point further, we may consider the pseudo eletric dipole moment (PEDM) encountered in ref. [31]. This was interpreted as a potential CPT-violating term in the amplitude under investigation. Actually, a careful analysis shows that, in a Minkowski background, the PEDM, if nonzero, would be imaginary and thus violate unitarity rather than CPT. Indeed, it was found to be zero only as a result of integrating over the moduli.

Finally we would like to stress that, since in the proof of the spacetime CPT theorem we did not need to specify in much detail how to evaluate the scattering amplitudes, our result can even be considered to be *non-perturbative* as far as one is able to give a non-perturbative interpretation of the string scattering amplitudes (2.21) which preserves all the assumptions of the theorem.

3. The physical interpretation of the space-time CPT transformation in string theory.

In the previous section we defined the CPT transformation by eqs. (2.22) and (2.23), and checked that it leads to the invariance (2.26) of the string amplitudes. But it still remains to be checked that the would-be CPT transformation has the correct space-time interpretation.

Indeed, taking the field theory limit ($\alpha' \to 0$), any string theory gives rise to a field theory whose spectrum is given by the massless spectrum of the string theory. This requires that the CPT transformation in string theory goes into the field theory one for the string

massless states.

We already know that the state $|\rho^{\mathrm{CPT}}\rangle$ is in the physical spectrum if and only if $|\rho\rangle$ is. We want to show that for any given single-string state $|\rho\rangle = |k, \eta, \{\lambda\}\rangle$, created by a vertex operator $\mathcal{W}_{|\rho\rangle}$, having helicity η and charges/labels $\{\lambda\}$, the CPT transformed state $|\rho^{\mathrm{CPT}}\rangle$ has helicity $-\eta$ and charges/labels $\{-\lambda\}$, i.e. is proportional to $|k, -\eta, \{-\lambda\}\rangle$. For mere labels the change of sign is to a certain extent a matter of convention, but for charges pertaining to a gauge symmetry group it is a necessity.

For notational simplicity we will restrict ourselves to the case $D = 4$ in what follows, but our arguments are readily generalized to other (even) values of D.

We start with a general discussion which we will then clarify with two explicit examples given in the following subsections.

In string theory any gauge charge λ (i.e. any weight pertaining to some generator of the Cartan subalgebra) is an eigenvalue of the zero mode operator Λ obtained from the corresponding hermitean Kač-Moody current, J_Λ, that involves only the "internal" CFT. That the string state has charge λ is equivalent to saying that the first order pole in the operator product expansion of J_Λ with $\mathcal{W}_{|\rho\rangle}$ is as follows:

$$J_\Lambda(z = \zeta_1)\mathcal{W}_{|\rho\rangle}(z = \zeta_2, \bar{z} = \bar{\zeta}_2) \overset{\mathrm{OPE}}{\sim} \frac{\lambda}{\zeta_1 - \zeta_2}\mathcal{W}_{|\rho\rangle}(z = \zeta_2, \bar{z} = \bar{\zeta}_2) . \tag{3.1}$$

We saw in the previous section that the space-time CPT transformation is essentially composed of the SR transformation and the world-sheet CPT transformation.

As is seen from the definition (1.28), the world-sheet CPT transformation changes sign on any chiral hermitean field of dimension one, in particular on any Kač-Moody current, and so it follows, by performing the world-sheet CPT transformation on both sides of eq. (3.1), that if the operator $\mathcal{W}_{|\rho\rangle}$ carries charge λ, then the operator $\mathcal{W}_{\langle\rho|} = \chi\widehat{\mathcal{W}}_{|\rho\rangle}$ carries charge $-\lambda$.

Since the SR transformation defined by eqs. (2.1), (2.5) and (2.9) only affects the space-time degrees of freedom, it is clear that if the operator $\mathcal{W}_{\langle\rho|}$ carries charge $-\lambda$, so does the operator $(\mathcal{W}_{\langle\rho|})^{\mathrm{SR}} = \mathcal{W}_{|\rho^{\mathrm{CPT}}\rangle}$, and hence the state $|\rho^{\mathrm{CPT}}\rangle$.

Thus we see that the sign change on λ is a result, not of the SR transformation, but of the world-sheet CPT transformation, i.e. of the map (2.20) relating the vertex operator $\mathcal{W}_{|\rho\rangle}$ and $\mathcal{W}_{\langle\rho|}$.

As regards the helicity, we define this to be the inner product of the angular momentum with a unit vector pointing in the same direction as the eigenvector of the four-momentum operator, as explained in greater detail in Appendix A. Under the world-sheet

CPT transformation relating $\mathcal{W}_{|\rho\rangle}$ to $\mathcal{W}_{\langle\rho|}$, both the momentum and the angular momentum in the direction of the momentum (being eigenvalues pertaining to the Kač-Moody currents of translations and rotations respectively) are flipped, meaning that the helicity is unaffected. Under SR the four-momentum operator flips sign but the angular momentum does not (since the position four-vector is also flipped); accordingly the operator $(\mathcal{W}_{\langle\rho|})^{\mathrm{SR}} = \mathcal{W}_{|\rho^{\mathrm{CPT}}\rangle}$ creates a state with helicity $-\eta$. In conclusion, the state $|\rho^{\mathrm{CPT}}\rangle$ has the opposite helicity than $|\rho\rangle$, as desired.

This concludes our proof that the map $|\rho\rangle \to |\rho^{\mathrm{CPT}}\rangle$ does indeed have the space-time properties of a CPT transformation. Notice that as such it automatically defines for us the concept of *anti-particle* in string theory. Simply, the CPT transformed of a string state $|\rho\rangle = |k, \eta, \{\lambda\}\rangle$ is (up to a phase) equal to the "anti-particle" string state of opposite helicity, $|\rho^{\mathrm{CPT}}\rangle \propto |k, -\eta, \{-\lambda\}\rangle$. Of course, the anti-particle state with the *same* helicity, i.e. $|k, +\eta, \{-\lambda\}\rangle$, needs not be in the physical spectrum. This depends on whether both signs of the helicity are allowed by the GSO projection conditions.

By construction the states $|\rho\rangle$ and $|\rho^{\mathrm{CPT}}\rangle$ have the same (bare) mass. Space-time CPT invariance guarantees that this equality holds also for the renormalized masses.

It is worthwhile to stress here the difference between world-sheet and space-time CPT. As was pointed out by Witten [32] the world-sheet CPT transformation relates a string state with given charge and mass to a string state with opposite charge and equal mass. So does the space-time CPT transformation we have defined. But whereas the world-sheet CPT transformation also changes sign on the energy of the string state, the space-time CPT transformation does not. Therefore, the space-time interpretation of the two transformations is very different: World-sheet CPT maps the vertex operator pertaining to an incoming string state into the vertex operator pertaining to the *same* string state but outgoing. Instead space-time CPT maps the vertex operator of an incoming "particle" string state into the vertex operator of an incoming "anti-particle" string state (of opposite helicity).

3.1   THE GLUON VERTEX OPERATOR.

In the first of our two examples we consider a "gluon", described in a $D = 4$ dimensional heterotic string theory by the vertex operator [22,31]

$$\mathcal{W}_{|\rho\rangle} = \frac{\kappa}{\pi} c\bar{c} \bar{J}^a \epsilon \cdot \psi e^{-\phi} e^{ik \cdot X} \ , \tag{3.2}$$

where κ is the gravitational coupling ($\kappa^2 = 8\pi G_N$ in $D = 4$), $\bar{J}^a$ is the hermitean Kac-Moody current, normalized so that $\bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) = \delta^{ab}(\bar{z} - \bar{w})^{-2} + \ldots$, and $e^{-\phi} = \delta(\gamma)$. For this picture the phase $\chi = +1$ in eq. (2.20) [22], and

$$\mathcal{W}_{|\rho\rangle} \xrightarrow{\text{CPT}} \mathcal{W}_{|\rho^{\text{CPT}}\rangle} = \left(\mathcal{W}_{\langle\rho|}\right)^{SR} = \left(\widehat{\mathcal{W}}_{|\rho\rangle}\right)^{SR} = \qquad (3.3)$$

$$= \left(\frac{\kappa}{\pi}e^{-ik\cdot X}e^{-\phi}\epsilon^* \cdot \psi \bar{J}^a \bar{c}c\right)^{SR}$$

$$= \left(\frac{\kappa}{\pi}c\bar{c}\bar{J}^a\epsilon^* \cdot \psi e^{-\phi}e^{-ik\cdot X}\right)^{SR}$$

$$= -\frac{\kappa}{\pi}c\bar{c}\bar{J}^a\epsilon^* \cdot \psi e^{-\phi}e^{ik\cdot X} \ .$$

If we assume ϵ to describe a photon of definite helicity (say, $\eta = +1$), then ϵ^* describes a photon of helicity $\eta = -1$ and thus the CPT transformed state is indeed proportional to the photon state with unchanged momentum but opposite helicity.

If instead we assume the photon to have linear polarization in a given direction, $\epsilon^\mu = \delta^\mu_\nu$, then CPT maps the state into minus itself. This is of course the correct behaviour well known from quantum field theory [21].

3.2 GROUND STATES OF WORLD-SHEET FREE FERMION (KLT) MODELS.

We now consider four-dimensional heterotic string models of the Kawai-Lewellen-Tye (KLT) type [14,15], where the internal degrees of freedom are described by 22 left-moving and 9 right-moving free complex fermions, which are labelled by $L = 1, \ldots, 22; 23, \ldots, 31$. In any given sector of the string theory a general state of the conformal field theory (excluding the reparametrization ghosts) can be built by means of non-zero mode creation operators acting on the states $|k, \mathbb{A}\rangle$ which are obtained by adding a nonzero space-time momentum k to the vacuum state(s) $|\mathbb{A}\rangle$ introduced in subsection 2.1.

If we bosonize all the 31 "internal" complex fermions (as well as the four Majorana fermions ψ^μ), using the explicit prescription for bosonization in Minkowski space-time proposed in ref. [26], the collective label $\mathbb{A}$ simply consists of the vacuum values of the "momenta" $J_0^{(L)} = \mathbb{A}_L$ pertaining to the 33 bosons $\Phi_{(L)}$ introduced by the bosonization, and the vacuum superghost charge $J_0^{(34)} = q = \mathbb{A}_{34}$ which is (minus) the "momentum" of the field $\phi \equiv \Phi_{(34)}$ that is introduced when "bosonizing" the superghosts. With the standard choice of picture, $q = -1$ for sectors of integer space-time spin and $q = -1/2$ for sectors of half odd integer space-time spin.

Since $[J_0^{(L)}, \Phi_{(K)}] = \delta^L{}_K$, the operator creating the state $|k, \mathbb{A}\rangle$ from the conformal vacuum is

$$S_{\mathbb{A}}(z, \bar{z}) \, e^{ik \cdot X(z, \bar{z})} \, , \tag{3.4}$$

where

$$S_{\mathbb{A}}(z, \bar{z}) \equiv \prod_{L=1}^{34} e^{\mathbb{A}_L \Phi_{(L)}(z, \bar{z})} \left(C_{(L)} \right)^{\mathbb{A}_L} \tag{3.5}$$

is a spin field operator and $C_{(L)}$ is a cocycle factor, see ref. [26] for details.

The values of the $\mathbb{A}_L$ in the vacuum (i.e. the values that minimize $L_0 + \bar{L}_0$) depend on the sector and hence on the details of the KLT model we happen to consider, see refs. [14,31]. Actually we may as well consider a slightly more general set of states, where the "internal" momenta $\mathbb{A}_L$, $L = 1, \ldots, 31$, assume any of the values allowed in the given sector, not just the values corresponding to the vacuum. But for the two "momenta" $\mathbb{A}_{32}$ and $\mathbb{A}_{33}$ pertaining to the bosonization of the ψ^μ we still restrict ourselves to the vacuum proper, meaning that in sectors of integer spin we have $\mathbb{A}_{32} = \mathbb{A}_{33} = 0$, so that the state $|\mathbb{A}\rangle$ is a space-time scalar, whereas in sectors of half odd integer spin we have $\mathbb{A}_{32} = \pm 1/2$ and $\mathbb{A}_{33} = \pm 1/2$, so that the state $|\mathbb{A}\rangle$ transforms as a space-time spinor, with the four-dimensional spinor index α is given by $\alpha = (\mathbb{A}_{32}, \mathbb{A}_{33})$.

We are interested in physical external states of this type, so we assume the level-matching condition $L_0 - \bar{L}_0 = 0$ to be satisfied and consider vertex operators given by

$$\mathcal{W}_{|k, \eta, \{\lambda\}\rangle}(z, \bar{z}) \;=\; W_{(in)}^{\mathbb{A}}(k, \eta, \{\lambda\}) \, S_{\mathbb{A}}(z, \bar{z}) \, e^{ik \cdot X(z, \bar{z})} c(z) \bar{c}(\bar{z}) \, . \tag{3.6}$$

Here the c-number quantities $W_{(in)}^{\mathbb{A}}(k, \eta, \lambda)$ are partially fixed by the GSO conditions. If the vertex operator in eq. (3.6) describes a state of spin $1/2$ ($\eta = \pm 1/2$), $W_{(in)}^{\mathbb{A}}(k, \eta, \{\lambda\})$ is a space-time spinor and has superghost charge $-1/2$, i.e. is proportional to $\delta_{\mathbb{A}_{34}, -1/2}$. It satisfies a Dirac equation which can be obtained by enforcing BRST invariance, more precisely, the requirement that the $3/2$-order pole in the operator product expansion (OPE) of the orbital part of the supercurrent, $T_F^{[X, \psi]}$, with the operator (3.6) should vanish. If we define the gamma matrices by the OPE

$$\psi^\mu(z) S_{\mathbb{A}}(w, \bar{w}) \stackrel{\text{OPE}}{=} \frac{1}{\sqrt{2}} \left(\boldsymbol{\Gamma}^\mu \right)_{\mathbb{A}}^{\mathbb{B}} S_{\mathbb{B}}(w, \bar{w}) \frac{1}{\sqrt{z - w}} + \ldots \, , \tag{3.7}$$

the Dirac equation assumes the matrix form

$$\left(W_{(in)}(k, \eta, \{\lambda\}) \right)^T \mathbb{D}(k) = 0 \qquad \text{or} \qquad \left(\mathbb{D}(k) \right)^T W_{(in)}(k, \eta, \{\lambda\}) = 0 \, , \tag{3.8}$$

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where the Dirac operator is

$$\mathbb{D}\left(k\right) = k_\mu \boldsymbol{\Gamma}^\mu - \mathbb{M} \ , \tag{3.9}$$

$\mathbb{M}$ being a mass operator that we do not need to write down explicitly.

Analogously, when the vertex operator in eq. (3.6) describes a space-time scalar ($\eta = 0$), BRST invariance is reduced to the requirement that the second order pole in the operator product expansion (OPE) of $T_F^{[X,\psi]}$ with the operator (3.6) vanishes. Depending on the details of the "internal" momenta this may be automatically satisfied; otherwise, it leads to an "internal" transversality condition on the $W_{(in)}^{\mathbb{A}}(k, \eta, \{\lambda\})$.

Consider first the case of a spin 1/2 string state. Under the space-time CPT transformation given by eq. (2.23), the vertex operator (3.6) transforms as follows

$$\mathcal{W}_{|k,\eta,\{\lambda\}\rangle}(z, \bar{z}) \xrightarrow{\mathrm{CPT}} \varphi_{SR}(W_{(in)}(k, \eta, \{\lambda\}))^\dagger \boldsymbol{\Gamma}^0 \mathbb{C}^{T,-1} \boldsymbol{\Gamma}_5 \, S(z, \bar{z}) \, e^{ik\cdot X(z,\bar{z})} c(z) \bar{c}(\bar{z}) \ , \tag{3.10}$$

where we used eq. (7.3) of ref. [22] and defined the charge conjugation matrix

$$\mathbb{C}_{\mathbb{AB}} = \left(\prod_{L=1}^{33} \delta_{\mathbb{A}_L + \mathbb{B}_L, 0} \right) \, \delta_{\mathbb{A}_{34}, \mathbb{B}_{34}} \, e^{i\pi \mathbb{A} \cdot Y \cdot \mathbb{B}} \ , \tag{3.11}$$

with Y being the 34×34 cocycle matrix Y_{KL} (see refs. [26,31]).

The state created by the CPT transformed vertex operator (3.10) has charges $\{-\lambda\}$ due to the presence of the charge conjugation matrix, and, by using the explicit form of the helicity operator described in Appendix A , has helicity $-\eta$. Thus it is a state of the form $|k, -\eta, \{-\lambda\}\rangle$, as it was required for the correct physical interpretation of the space-time CPT transformation.

Analogous considerations can be done for space-time scalars and can be generalized to any other vertex operator in the theory.

4.   The Spin-Statistics Relation

One of the assumptions on which our proof of CPT invariance relies is that for a physical string state, the statistics of the vertex operator on the world-sheet should be given by the space-time spin of the string state, in accordance with the well-known spin-statistics theorem of quantum field theory [17,6,18]. More precisely, we required that two vertex operators (describing physical string states that can appear together in a nonzero

amplitude) anti-commute if both carry half odd integer space-time spin and commute otherwise.

In field theory in more than two dimensions, the spin statistics theorem is well understood. It follows directly from the basic axioms of field theory and the proof is quite parallel and related to the one of the CPT theorem itself [6]. One way of looking at it is to say that the spin statistics theorem is one of the steps required in the proof of the CPT theorem.

In field theory the spin statistics theorem requires that a field of half odd integer spin is quantized according to canonical anti-commutation relations and a field of integer spin according to canonical commutation relations. An equivalent formulation is that the free-field contribution to the vacuum energy is positive (negative) for fields of integer (half odd integer) spin. In its more general form the theorem does *not* require a given fermionic field to necessarily anti-commute with *other* fermionic fields or a bosonic field to necessarily commute with all other fields, whether bosonic or fermionic. The situation where some field has abnormal statistics with respect to some of the other fields is referred to as *para-statistics*. However, one may show [33] that in more than two dimensions there always exists a map to a new basis of fields (a so-called Klein map [34]) which gives an equivalent physical system where all fields obey the standard spin-statistics relation [6].

In string theory the situation is not so well understood. We do not have an axiomatic framework (such as a fully-fledged string field theory) from which we could derive general theorems, this is why we had to prove the space-time CPT theorem directly on the scattering amplitudes. So, if we do not want to simply assume the spin-statistics relation and discard any string model which does not satisfy it, the approach that we have taken up to now in this paper, all what we can do is to discuss how the spin-statistics relation appears in string theory. We do not have major results to report and in this section we will just try to illustrate a few points on this subject.

In building a Neveu-Schwarz Ramond-like string model, various consistency requirements, in particular modular invariance, forces one to impose a sensible projection on the spectrum, the GSO projection [GSO].

In all known cases the GSO projections thus constructed also happen to select the physical states in such a way that states of integer space-time spin contribute to the one-loop partition function with an overall plus sign, whereas states of half-odd integer space-time spin contribute with an overall minus sign [seiberg-witten,Anto]. Thus, all known

consistent models require a GSO projection which implies a space-time spin-statistics relation. As far as we know, a proof from first principles that this is the only possibility, i.e. of the absence of pathological string theories violating the spin-statistics relation, is not known.

Thus our expectation is that the GSO conditions should be crucial for obtaining the desired relation (2.24) between space-time spin and world-sheet statistics of the physical state vertex operators. On the other hand, we do not expect the GSO conditions *alone* to guarantee such a relation. One might imagine having para-statistics also in string theory, i.e. vertex operators pertaining to *different* string states might have unusual statistics with respect to one another.

Indeed, we can always take the field theory limit of a string model thus obtaining a field theory whose spectrum is composed by the massless modes of the string spectrum. Since para-statistics is an a priori possibility in field theory, this argument, although it does not constitute a proof, strongly suggests the possibility of para-statistics also in string theory.

So, we would like to understand if it is possible to build string models satisfying para-statistics and, in this case, if there always exists a physically equivalent string model satisfying the standard statistics. Should string models exist where this is not the case, the proof we have given of the CPT theorem would break down for such models.

We do not have a general answer to this question, but we can study what happens in a particular class of string theories.

We restrict ourselves to the KLT models, that is, four dimensional heterotic string models built with free world-sheet fermions [14,15,16]. As in subsection 3.2 we bosonize all fermions. We want to compute the relative statistics of any two physical vertex operators in the model, i.e. find the phase that appears when the two operators are interchanged. We may ignore the universal factor $c\bar{c}\exp\{ik\cdot X\}$ which is always commuting. In fact, it is enough to consider string states created by the spin field operator $S_{\mathbb{A}}$, given by eq. (3.5), assuming that we let the $\mathbb{A}_L$, $L = 1,\ldots,33$, take *any* of the values allowed in the string model (i.e. do not restrict ourselves to the vacuum values in each sector). This is because all other states are obtained by means of the non-zero-mode creation operators of the fields $\Phi_{(L)}$, which are all commuting and do not affect either the spin (which is encoded in the values $\mathbb{A}_{32}$ and $\mathbb{A}_{33}$) or the GSO conditions, which only involve the fermion numbers, i.e. the values of $\mathbb{A}_L$, $L = 1,\ldots,34$. As always $\mathbb{A}_{34} = -1$ $(-1/2)$ for sectors of integer (half odd integer) spin, and no superghost excitations above this vacuum are allowed for physical

states.

By using the conventions of refs. [26,22,31], we can easily compute the phase one obtains when transposing two generic spin field operators in a KLT model. We get

$$S_{\mathbb{A}_1} S_{\mathbb{A}_2} = e^{i\pi \sum_{K,L} \mathbb{A}_{1,L} \widetilde{Y}_{L,K} \mathbb{A}_{2,K}} S_{\mathbb{A}_2} S_{\mathbb{A}_1} = e^{i\pi \mathbb{A}_1 \cdot \widetilde{Y} \cdot \mathbb{A}_2} S_{\mathbb{A}_2} S_{\mathbb{A}_1} , \qquad (4.1)$$

where the 34×34 matrix $\widetilde{Y}$ has all elements equal to $+1$ or -1 and is obtained by anti-symmetrizing the lower-triangular matrix Y, which describes the choice of cocycles, and adding the diagonal, $\mathrm{diag}\widetilde{Y}$, equal to $-\epsilon$ ($= \pm 1$) for the first 22 entries as well as the 34th, and equal to $+\epsilon$ for the entries from the 23rd to the 33rd. The matrix $\widetilde{Y}$ is subject to the constraints described in refs. [31,26] which ensure that the BRST current has well-defined statistics and that the Picture Changing operator, as well as all Kač-Moody currents, have bosonic statistics w.r.t. any of the spin fields $S_{\mathbb{A}}$.

Thus everything depends on the phase

$$\phi[\mathbb{A}_1, \mathbb{A}_2] \equiv \sum_{K,L} \mathbb{A}_{1,L} \widetilde{Y}_{L,K} \mathbb{A}_{2,K} = \mathbb{A}_1 \cdot \widetilde{Y} \cdot \mathbb{A}_2 , \qquad (4.2)$$

which a priori depends on the choice of cocycles as well as on the two states we are considering.

First we want to verify that the operator $S_{\mathbb{A}}$ satisfies Bose or Fermi statistics with respect to itself, depending only on whether the space-time spin is integer or half-odd integer. Thus we assume that $\mathbb{A}_1 = \mathbb{A}_2 = \mathbb{A}$. Recalling that $\widetilde{Y}_{K,L} = -\widetilde{Y}_{L,K}$ for $K \neq L$ we can rewrite eq. (4.2) as

$$\phi[\mathbb{A}, \mathbb{A}] = \sum_L \mathbb{A}_L \widetilde{Y}_{LL} \mathbb{A}_L = 2\epsilon \left(-\frac{1}{2} \sum_{L=1}^{22} (\mathbb{A}_L)^2 - \frac{1}{2}(\mathbb{A}_{34})^2 + \frac{1}{2} \sum_{L=1}^{33} (\mathbb{A}_L)^2 \right)$$
$$= 2\epsilon \left(L_0[\mathbb{A}] - \bar{L}_0[\mathbb{A}] - L_0^{(\beta\gamma)}[\mathbb{A}_{34}] - \tfrac{1}{2}(\mathbb{A}_{34})^2 \right) . \qquad (4.3)$$

Since all non-zero-mode creation operators in our bosonized formulation carry integer values of L_0 and $\bar{L}_0$, level matching implies that $L_0[\mathbb{A}] - \bar{L}_0[\mathbb{A}] =$ integer, and since the phase $\phi[\mathbb{A}_1, \mathbb{A}_2]$ is obviously defined modulus 2, the first two terms cancel and we remain with

$$\phi[\mathbb{A}, \mathbb{A}] \stackrel{\mathrm{MOD \ 2}}{=} -2\epsilon \left(L_0^{(\beta\gamma)}[\mathbb{A}_{34}] + \tfrac{1}{2}(\mathbb{A}_{34})^2 \right) . \qquad (4.4)$$

Now it is enough to remember that the energy of the superghost system is $L_0^{(\beta\gamma)}[\mathbb{A}_{34}] = -\frac{1}{2}\mathbb{A}_{34}(\mathbb{A}_{34} + 2)$ to obtain

$$\phi[\mathbb{A}, \mathbb{A}] \stackrel{\mathrm{MOD \ 2}}{=} 2\epsilon \, \mathbb{A}_{34} \stackrel{\mathrm{MOD \ 2}}{=} \begin{cases} 0 & \text{in a sector of integer spin} \\ 1 & \text{in a sector of half odd integer spin} \end{cases} . \qquad (4.5)$$

Thus level-matching alone leads to the correct spin-statistics relation of the operator $S_\mathbb{A}$ with itself.

Now we discuss the case of two different spin fields. The analysis for a general four-dimensional KLT model is possible but technically very complicated, and since our aim is merely to illustrate the issues involved by means of an example, we restrict ourselves to the simplest possible model in four dimensions, where all world-sheet fermions have the same spin structure, which can be either Ramond (R) or Neveu-Schwarz (NS). In this simple model there is only one GSO condition, which can be written on the form

$$\mathbf{V}_0 \cdot G \cdot \mathbb{A} \;\overset{\text{MOD } 1}{=}\; \begin{cases} 0 & \text{in the NS sector} \\ 1/2 + k_{00} & \text{in the R sector} \end{cases}. \tag{4.6}$$

Here $\mathbf{V}_0$ is a 34-vector with all entries equal to $1/2$, G is a diagonal 34×34 matrix given by

$$G \;=\; \text{diag}\left((+1)^{22}; (-1)^{11}; +1\right) \;=\; -\epsilon \, \text{diag} \widetilde{Y} \tag{4.7}$$

and k_{00} is a free parameter that equals either 0 or $1/2$ (mod 1).

Obviously we have three cases to consider: i) both vertex operators are in the NS sector and describe states of integer spin, ii) both vertex operators are in the R sector and describe states of half odd integer spin, iii) one vertex operator is in the NS sector and one in the R sector, describing then one state of integer and one of half odd integer spin.

In case i) both $\mathbb{A}_{1,L}$ and $\mathbb{A}_{2,K}$ are integer so that

$$\phi[\mathbb{A}_1, \mathbb{A}_2] \;=\; \sum_{K,L} \mathbb{A}_{1,L} \widetilde{Y}_{L,K} \mathbb{A}_{2,K} \;\overset{\text{MOD } 2}{=}\; \sum_{K,L} \mathbb{A}_{1,L} \mathbb{A}_{2,K} \tag{4.8}$$

$$\overset{\text{MOD } 2}{=}\; \left(\sum_L G_{LL} \mathbb{A}_{1,L} \right) \left(\sum_K G_{KK} \mathbb{A}_{2,K} \right) \;=\; \left(2\mathbf{V}_0 \cdot G \cdot \mathbb{A}_1\right)\left(2\mathbf{V}_0 \cdot G \cdot \mathbb{A}_2\right) \;.$$

We see that exactly because of the GSO condition (4.6), $\phi[\mathbb{A}_1, \mathbb{A}_2] \overset{\text{MOD } 2}{=} 0$ in this case, meaning that regardless of the choice of cocycles the two vertex operators commute, in agreement with standard statistics.

Consider now the case iii), where $S_{\mathbb{A}_1}$ describes a state of integer spin and $S_{\mathbb{A}_2}$ one of half odd integer spin. In this case $\mathbb{A}_{1,L}$ is still integer, whereas the $\mathbb{A}_{2,K}$ are half odd integer. We rewrite

$$\mathbb{A}_2 = (\mathbb{A}_2 - \mathbf{V}_0) + \mathbf{V}_0 \;, \tag{4.9}$$

where now $\mathbb{A}_2 - \mathbf{V}_0$ is a vector of integers, and by proceeding as before we find

$$\phi[\mathbb{A}_1, \mathbb{A}_2] \;\overset{\text{MOD } 2}{=}\; \left(2\mathbf{V}_0 \cdot G \cdot \mathbb{A}_1\right)\left(2\mathbf{V}_0 \cdot G \cdot (\mathbb{A}_2 - \mathbf{V}_0)\right) \;+\; 2\mathbb{A}_1 \cdot \widetilde{\mathbf{V}}_0 \;, \tag{4.10}$$

where we introduced the auxiliary 34-component vector (see eq. (2.43) of ref. [31])

$$(\widetilde{\mathbf{V}}_0)_L \;\equiv\; \tfrac{1}{2}\sum_K \widetilde{Y}_{LK}(\mathbf{V}_0)_K \;, \tag{4.11}$$

which by construction has entries that are 0 or 1/2 (mod 1). Now, the first term on the right-hand side of (4.10) vanishes, since $2\mathbf{V}_0 \cdot G \cdot (\mathbb{A}_2 - \mathbf{V}_0)$ is manifestly integer, and $2\mathbf{V}_0 \cdot G \cdot \mathbb{A}_1 \stackrel{\mathrm{MOD}\ 2}{=} 0$ by the GSO condition (4.6). But the second term, $2\mathbb{A}_1 \cdot \widetilde{\mathbf{V}}_0$, can a priori be both an even or an odd integer, depending on the choice made for the cocycle matrix and the values taken for the $\mathbb{A}_{1,L}$. Thus we do indeed have para-statistics in this string model *unless* we arrange the cocycle matrix in such a way that

$$\widetilde{\mathbf{V}}_0 \stackrel{\mathrm{MOD}\ 1}{=} G \cdot \mathbf{V}_0 \stackrel{\mathrm{MOD}\ 1}{=} \mathbf{V}_0 \;. \tag{4.12}$$

Then and only then will the second term on the right-hand side of (4.10) vanish for all possible values of $\mathbb{A}_{1,L}$, by virtue of the GSO condition (4.6).

It is interesting to note that the cocycle consistency conditions described in detail in ref. [31] allow the vector $\widetilde{\mathbf{V}}_0$ to take any one of the following four forms (mod 1),

$$\left((1/2)^{34}\right) \qquad \left((1/2)^{22}(0)^{12}\right) \qquad \left((0)^{22}(1/2)^{12}\right) \qquad \left((0)^{34}\right) \;, \tag{4.13}$$

but only the first of the four choices (i.e. $\widetilde{\mathbf{V}}_0 \stackrel{\mathrm{MOD}\ 1}{=} \mathbf{V}_0$) leads to standard statistics between physical string states residing in the R and the NS sector, whereas the other three choices lead to para-statistics.

We still have to check that a cocycle choice satisfying (4.12) also leads to standard statistics in the case ii), when both $S_{\mathbb{A}_1}$ and $S_{\mathbb{A}_2}$ reside in the R sector. By decomposing both $\mathbb{A}_1$ and $\mathbb{A}_2$ according to (4.9) we find in this case

$$\phi[\mathbb{A}_1,\mathbb{A}_2] \stackrel{\mathrm{MOD}\ 2}{=} \; (2\mathbf{V}_0 \cdot G \cdot (\mathbb{A}_1 - \mathbf{V}_0))\,(2\mathbf{V}_0 \cdot G \cdot (\mathbb{A}_2 - \mathbf{V}_0)) \tag{4.14}$$
$$+\; \mathbf{V}_0 \cdot \widetilde{Y} \cdot (\mathbb{A}_2 - \mathbf{V}_0) \;+\; (\mathbb{A}_1 - \mathbf{V}_0) \cdot \widetilde{Y} \cdot \mathbf{V}_0 \;+\; \mathbf{V}_0 \cdot \widetilde{Y} \cdot \mathbf{V}_0 \;.$$

By using the GSO conditions, the first term on the right hand side becomes $(1 + 2k_{00})^2$ mod 2. The second term we rewrite as

$$\mathbf{V}_0 \cdot \widetilde{Y} \cdot (\mathbb{A}_2 - \mathbf{V}_0) \;=\; -(\mathbb{A}_2 - \mathbf{V}_0) \cdot \widetilde{Y} \cdot \mathbf{V}_0 \;+\; 2\mathbf{V}_0 \cdot \mathrm{diag}\widetilde{Y} \cdot (\mathbb{A}_2 - \mathbf{V}_0) \tag{4.15}$$
$$\stackrel{\mathrm{MOD}\ 2}{=} \; -2(\mathbb{A}_2 - \mathbf{V}_0) \cdot G \cdot \mathbf{V}_0 \;-\; 2\epsilon\mathbf{V}_0 \cdot G \cdot (\mathbb{A}_2 - \mathbf{V}_0) \stackrel{\mathrm{MOD}\ 2}{=} 0 \;.$$

The third term becomes

$$(\mathbb{A}_1 - \mathbf{V}_0) \cdot \widetilde{Y} \cdot \mathbf{V}_0 \stackrel{\text{MOD } 2}{=} 2(\mathbb{A}_1 - \mathbf{V}_0) \cdot G \cdot \mathbf{V}_0 \stackrel{\text{MOD } 2}{=} 1 + 2k_{00} , \qquad (4.16)$$

which cancels the first term mod 2, since $1 + 2k_{00}$ is an integer. And finally

$$\mathbf{V}_0 \cdot \widetilde{Y} \cdot \mathbf{V}_0 = \mathbf{V}_0 \cdot \text{diag} \widetilde{Y} \cdot \mathbf{V}_0 \stackrel{\text{MOD } 2}{=} 1 . \qquad (4.17)$$

Thus, a cocycle choice consistent with eq. (4.12) does indeed lead to ordinary statistics between any pair of physical vertex operators.

One might still wonder whether a cocycle choice exists which satisfies eq. (4.12). There is no need to worry: A simple analysis shows that there exist 2^{529} such choices!

To conclude, we have seen that already in the simplest conceivable four-dimensional KLT model, a generic choice of cocycles, even one subject to the constraints of ref. [31], gives us vertex operators which satisfy *para-statistics* in the sense of ref. [6]. But there exists a class of cocycle choices for which the vertex operators satisfy the standard statistics.

Since we believe that, once the requirements of ref. [31] are satisfied, the physics should not depend on the specific choice of cocycles, we can interpret our result as the proof that, at least in this very simple model, the situations leading to para-statistics are physically equivalent to one where all vertex operators satisfy the standard statistics.

The above analysis can be extended to the case of general four-dimensional heterotic KLT string models, but a full discussion is technically quite complicated and will be reported in a future publication.

5.　　Conclusions and outlook

In this final section we would like to make a few comments on what we have seen so far.

We have introduced a world-sheet transformation applicable to any string theory in a D dimensional (D even) Minkowski background which gives rise to the space-time CPT transformation when acting on vertex operators of physical string states. We have also shown that under the hypothesis of Lorentz invariance and spin-statistics the CPT theorem holds, at least to any order in perturbation theory.

That being the case, any breaking of CPT invariance in such a string theory would have to be due either to subtle non-perturbative effects, like those proposed by Ellis et al. [2], or due to a spontaneous move away from the original background, in the manner of the mechanism suggested by Kostelecky and Potting [1], where some field in the low-energy effective field theory which is odd under CPT (for example a vector field) is imagined to acquire a nonzero vacuum expectation value. This mechanism naturally leads us to consider to which extent it is possible to generalize the CPT theorem to general backgrounds. The answer to this question might be more easily perceived if we could find a proof of the CPT theorem which does not use so directly the hypothesis of Lorentz invariance, but relies more on just the world-sheet properties of the strings.

The relation between space-time spin and world-sheet statistics of physical state vertex operators is another point that deserves a more thorough investigation. This obviously brings up the issue of what are the properties satisfied by a string theory in analogy with the Wightman axioms in field theory, or in other words, which are the *minimum* assumptions that we can make in trying to define and prove the spin-statistics theorem or the CPT theorem in string theory.

It is clear that a full understanding of the role of CPT invariance and spin-statistics in string theory is still lacking. Perhaps we will have to await an improved understanding of the nature of string theory itself.

Appendix A: Helicity in String Theory

In this appendix we give our conventions for helicity and show in detail how the helicity sign change under CPT comes about in an explicit example.

We define the helicity as the inner product of the angular momentum $\vec{J}$ with a unit vector pointing in the direction of the momentum,

$$\eta = \frac{\vec{J} \cdot \vec{k}}{|\vec{k}|} = \frac{1}{2} \epsilon_{ijk} M^{jk} k^i \frac{1}{|\vec{k}|} \; , \tag{A.1}$$

where the generators $M^{\mu\nu}$ of the Lorentz group are given by

$$M^{\mu\nu} = \oint_0 \frac{\mathrm{d}z}{2\pi i} \left[\frac{-1}{2i} \left(X^\mu \partial_z X^\nu - X^\nu \partial_z X^\mu \right) - i \psi^\mu \psi^\nu \right] \tag{A.2}$$

$$+ \oint_0 \frac{\mathrm{d}\bar{z}}{2\pi i} \left[\frac{-1}{2i} \left(X^\mu \partial_{\bar{z}} X^\nu - X^\nu \partial_{\bar{z}} X^\mu \right) \right] \; .$$

In section 3 we argued on general grounds that the helicity is flipped by the space-time CPT transformation. As an example, consider the physical vertex operator (3.6) in the case where this describes a string state of spin $1/2$. Acting on a space-time spinorial state $|k, \mathbb{A}\rangle$ created from the conformal vacuum by means of the operator (3.4),

$$M^{\mu\nu} \, |k, \mathbb{A}\rangle \;=\; \left\{ (q^\mu k^\nu - q^\nu k^\mu)\delta_\mathbb{A}^{\;\mathbb{B}} \;+\; \frac{i}{4} \left[\boldsymbol{\Gamma}^\mu, \boldsymbol{\Gamma}^\nu \right]_\mathbb{A}^{\;\mathbb{B}} \right\} \, |k, \mathbb{B}\rangle \;, \qquad (A.3)$$

where q^μ is the operator conjugate to k^ν and the gamma matrices are defined by eq. (3.7).

Therefore, in order for the physical state created by the vertex operator (3.6) to have helicity η, the space-time spinor $W_{(in)}^\mathbb{A}(k, \eta, \{\lambda\})$ should satisfy the matrix eigenvalue equation

$$\left(W_{(in)}(k, \eta, \{\lambda\}) \right)^T \left(\frac{i}{8}\epsilon_{ijk} \frac{k^i}{|\vec{k}|} \left[\boldsymbol{\Gamma}^j, \boldsymbol{\Gamma}^k \right] - \eta \right) \;=\; 0 \;. \qquad (A.4)$$

Assuming this to be the case we would like to verify that the CPT transformed vertex operator (3.10) has helicity $-\eta$, i.e. that

$$\left(W_{(in)}(k, \eta, \{\lambda\}) \right)^\dagger \boldsymbol{\Gamma}^0 \mathbb{C}^{T,-1} \boldsymbol{\Gamma}_5 \left(\frac{i}{8}\epsilon_{ijk} \frac{k^i}{|\vec{k}|} \left[\boldsymbol{\Gamma}^j, \boldsymbol{\Gamma}^k \right] + \eta \right) \;=\; 0 \;. \qquad (A.5)$$

This follows immediately if we take the complex conjugate of eq. (A.4) and multiply from the right by $\boldsymbol{\Gamma}^0 \mathbb{C}^{T,-1} \boldsymbol{\Gamma}^5$, using the properties

$$\begin{aligned} \boldsymbol{\Gamma}^0 \boldsymbol{\Gamma}^\mu \;&=\; -\left(\boldsymbol{\Gamma}^\mu \right)^\dagger \boldsymbol{\Gamma}^0 \;, & (A.6) \\ \left\{ \boldsymbol{\Gamma}^5, \boldsymbol{\Gamma}^\mu \right\} \;&=\; 0 \;, & \\ \boldsymbol{\Gamma}^\mu \boldsymbol{\Gamma}^\nu \mathbb{C} \;&=\; \mathbb{C} \left(\boldsymbol{\Gamma}^\mu \right)^T \left(\boldsymbol{\Gamma}^\nu \right)^T \;. & \end{aligned}$$

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